Technical Notes

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Influence of the Pressure Gradient on the Law of the Wall

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Nomenclature

C	= additive constant of the logarithmic law of the wall
p	= static pressure
p^*	= dimensionless pressure gradient, $\nu/\rho u_\tau^3 (\mathrm{d}p/\mathrm{d}x)$
Re_U	= Reynolds number based upon the width of the channel and the maximum value of the velocity
r	= cylindrical polar coordinate in radial direction
u	= mean velocity component parallel to the wall
u_τ	= friction velocity, $(\tau_w/\rho)^{1/2}$
u^*	= nondimensional u velocity, u/u_τ
x	= longitudinal coordinate parallel to the wall
y	= perpendicular coordinate normal to the wall
y^*	= nondimensional wall distance, yu_τ/ν
κ	= von Kármán constant
ν	= kinematic viscosity
$-\rho\overline{u'v'}$	= Reynolds shear stress
ρ	= fluid density
τ	= total shear stress, $\tau = \tau_\ell + (-\rho\overline{u'v'})$
τ_ℓ	= viscous shear stress
τ_w	= wall shear stress

Introduction

IN his dissertation, Coantic[1] lists about 40 laws of the wall. However, only nine of these describe in one single expression the entire inner velocity profile of a boundary flow. Among five of these laws, presented in implicit or parametric form, is Spalding's equation.[2] This equation may be regarded as the most familiar one. Of four of these nine laws given in explicit fashion, van Driest's[3] law of the wall, which is in integral form, should be mentioned. Spalding, van Driest, and other papers cited by Coantic refer in their laws of the wall to the boundary layer of the flat plate with constant pressure in the main direction of flow.

The main effect of the pressure gradient on the velocity profile is investigated, e.g., by Szablewski[4] and Townsend,[5] and leads to laws of the wall that are difficult to operate in further analytical work. The distribution of the shear stress is usually substituted by Prandtl's "hypothesis of mixing length." The results of these considerations depend upon the quality and applicability of the shear stress approach used.

In the present paper a law of the wall is introduced which in a simple algebraic expression describes the entire inner velocity distribution of the boundary-layer flow in a fully developed turbulent channel. At the wall as well as in the immediate vicinity of the wall, the pressure forces balance the viscosity forces. This circumstance and the dependence, found by Reichardt[6] for this region of the turbulent shear stress on the wall distance has to be taken into account in the law of the wall. The other boundary condition is imposed by the logarithmic law of the wall. It is to be assumed that the pressure gradient also has an effect in this region. However, the investigation aims mainly at the immediate wall region where the physical boundary conditions can be clearly formulated.

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Law of the Wall and Shear Stress Distribution

Preliminary Considerations

The mean time velocity distributions of a fully developed turbulent channel flow are congruent. Thus, neglecting the changes in longitudinal and transversal fluctuations in the direction of flow x, one obtains the Reynolds equation

$$0 = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{1}{\rho}\frac{\delta\tau}{\delta y} \tag{1}$$

By replacing the total shear stress in this equation with its viscous and turbulent parts and then dividing the entire relation by the wall shear stress where, after integration over the dimensionless wall distance y^*, the Reynolds shear stress results in

$$-\frac{\rho\overline{u'v'}}{\tau_w} = 1 + p^*y^* - \frac{\delta u^*}{\delta y^*} \tag{2}$$

According to Reichardt the turbulent shear stress in the immediate vicinity of the wall has to be at least proportional to y^3, the above equation may be written

$$\Theta(y^{*3}) \sim 1 + p^*y^* - \frac{\delta u^*}{\delta y^*} \tag{3}$$

Integrating this relation, the following dependence for the velocity distribution very close to the wall can be formulated.

$$u^* \sim y^* + \tfrac{1}{2}p^*y^{*2} + \Theta(y^{*4}) \tag{4}$$

The fully turbulent pipe flow under the same conditions as those for the channel flow results from the equation of motion in cylinder coordinates parallel to the pipe axis

$$0 = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} - \frac{1}{r}\frac{\delta}{\delta r}(r\overline{u'v'}) + \nu\left(\frac{\delta^2 u}{\delta r^2} + \frac{1}{r}\frac{\delta u}{\delta r}\right) \tag{5}$$

Continuing this analysis and also regarding the boundary conditions, the following proportionality for the boundary profile in the immediate vicinity of the wall is valid

$$u^* \sim y^* + \tfrac{1}{4}p^*y^{*2} + \Theta(y^{*4}) \tag{6}$$

Therefore the wall distance is the difference between the radius of the pipe and the radial distance to the pipe axis r.

Deduction of the Law of the Wall

For a fully developed turbulent channel flow, the velocity distribution resulting from the viscous sublayer, the transition layer, and the logarithmic region can be described by

$$u^* = \frac{1}{\kappa}\ln(1+a_1 y^*) + C_1\left[1-e^{-a_2 y^*}(1+a_3 y^*)\right] \tag{7}$$

The following equations are obtained by replacing $(1+a_1y^*)^{-1}$ and the exponential function in the differentiated Eq. (7) with the power series and inserting it in Eq. (2). At the same time care must be taken that in the turbulent shear stress no power smaller than the third in y^* may appear

$$0 = y^{*0}\left[1-\frac{a_1}{\kappa}-C_1(a_2-a_3)\right] \tag{8}$$

$$0 = y^{*1}\left[p^*+\frac{a_1^2}{\kappa}+C_1 a_2(a_2-2a_3)\right] \tag{9}$$

$$0 = y^{*2}\left[\frac{a_1^3}{\kappa}+\frac{1}{2}C_1 a_2^2(a_2-3a_3)\right] \tag{10}$$

The fourth equation needed results from the demand that for greater wall distances the values of the new law of the wall have to merge with those of the logarithmic law. This is fulfilled in Eq. (11)

$$C - C_1 - (1/\kappa)\ln a_1 = 0 \tag{11}$$

Table 1 lists the newly computed constants, where p^* is taken from channel measurements made by Eckelmann[7] with $\kappa = 0.4$ and $C = 5.5$

As an example of a complete set of measurements for the flat plate as far as and including the viscous sublayer, Eq. (7) is compared with experimental results from Liu, Kline, and Johnston[8] in Fig. 1. In this case quantities a_1, a_2, a_3, and C_1 become constant values for the given κ and C. Using the values given by Nikuradse[9] and Coles and Hirst[10] the new constants are determined as shown in Table 2.

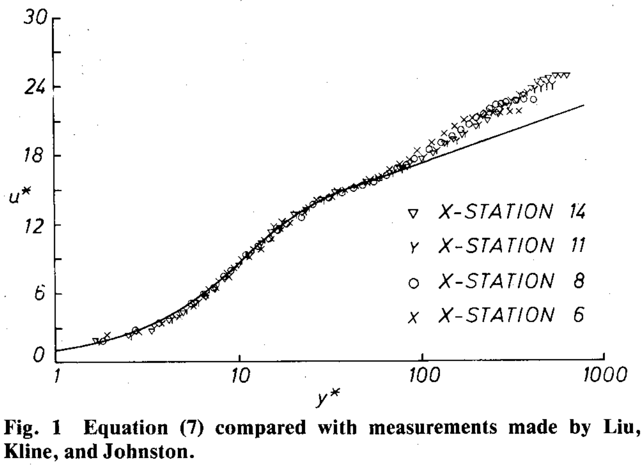

Fig. 1 Equation (7) compared with measurements made by Liu, Kline, and Johnston.

Table 1 a_1, a_2, a_3, and C_1 depending on p^*

Re_U	$p^* \times 10^3$	a_1	a_2	a_3	C_1
5600	−7	0.215	0.168	0.118	9.347
8200	−4.8	0.218	0.17	0.121	9.306

Table 2 New constants for the flat plate

Source	κ	C	a_1	a_2	a_3	C_1
Ref. 9	0.4	5.5	0.226	0.174	0.127	9.22
Ref. 10	0.41	5.0	0.238	0.186	0.137	8.497

Distribution of the Reynolds Shear Stress

In a fully developed turbulent channel flow the Reynolds shear stress has been determined from Eq. (2) using the velocity approach of Eq. (7), and the results are compared in Fig. 2 with the values given by Eckelmann. In case I the values for a_1, a_2, a_3, and C_1 have been taken from Table 1, which take into account the effect of the pressure gradient in the immediate vicinity of the wall. For case II the constants of the flat plate, with $dp/dx = 0$, have been used in the velocity approach. Whereas in case I the distribution of the turbulent shear stress satisfies the boundary conditions, in case II one obtains a decrease of $-\rho\overline{u'v'}$ in the immediate vicinity of the wall and at the same time negative Reynolds shear stress values, which is physically untenable. Directly at the wall the gradient of curve II is identical to p^*.

Discussions of the Constants κ and C

All of the considerations produced so far concerning the effect of the pressure gradient on the law of the wall are valid only in the immediate vicinity of the wall.

The considerations would have been incomplete if the constants κ and C had not been mentioned. Among others, Patel and Head[11] discuss fully turbulent channel and pipe flows where C is variable and κ is constant. In Ref. 12 the correlation between κ and C is stated. With the aid of fully turbulent pipe flows of small Reynolds numbers the velocity distributions thus calculated are compared with measurement data. According to Ref. 13, for general turbulent boundary layers it is possible to calculate κ and C depending on p^* and the gradient of the dimensionless wall shear stress, e.g., by accelerated flow. Heretofore boundary-layer profiles must be apparently parallel to measured $\overline{u'v'}$ distributions. Pfeil and Göing[14] suggest another method that makes a fitted u_τ from the logarithmic law of the wall by making constant κ and C responsible for the discrepancy in the right and left sides of the equation of momentum. Therefore a new u_τ is obtained from the momentum equation and so new κ and C values are dependent on the pressure gradient rendered dimensionless by the momentum thickness.

Summary

Taking a fully developed turbulent channel flow as an example shows that in the region close to the wall the velocity profile of a boundary-layer flow is not only a function of the dimensionless wall distance, but also depends upon the pressure gradient. This is required by the equation of motion directly at the wall as well as by the distribution of the Reynolds shear stress, as found by Reichardt from the equation of continuity in the immediate vicinity of the wall. Therefore a law of the wall is introduced here, the constants of which depend not only on the universal constants present in the logarithmic law of the wall, but also on p^*. This law is

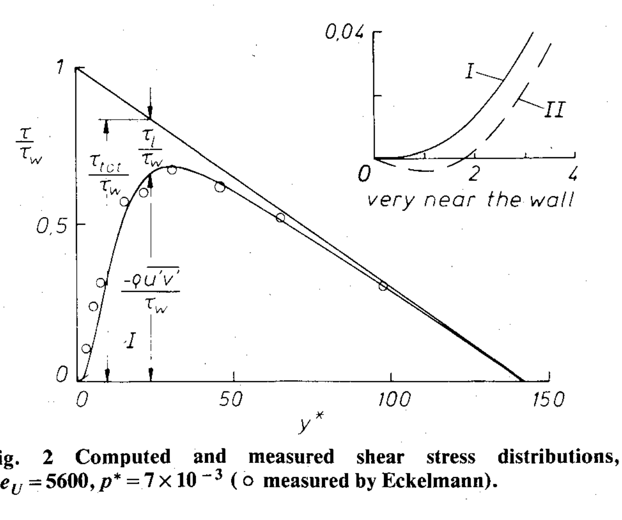

Fig. 2 Computed and measured shear stress distributions, $Re_U = 5600$, $p^* = 7\times10^{-3}$ (o measured by Eckelmann).

also applicable for cylindrical coordinates; in this case only the coefficient of p^* has to be changed in the system of equations.

The considerations in the logarithmic region do not directly concern the law of the wall presented here, since new findings in this area can be easily processed mathematically. The authors' suggestions concerning this problem are added to stimulate discussion of the universality of the constants in the logarithmic law of the wall.

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A Generalized Algebraic Stress Transport Hypothesis

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SOME 10 years ago, Rodi[1] and this writer[2,3] independently proposed ways of truncating a second-moment (or Reynolds stress) closure so that the differential transport equations for the Reynolds stresses $\overline{u_i u_j}$ were replaced by a single transport equation for the turbulent kinetic energy k $(\equiv \overline{u_i u_i}/2)$ and a set of algebraic equations for the Reynolds stresses. Such closures have become widely used and are commonly known as "algebraic stress models." The form adopted in Refs. 2 and 3 implies that the net transport of $\overline{u_i u_j}$ denoted by T_{ij} (i.e., convection C_{ij} minus diffusion D_{ij}) is related to the turbulent kinetic energy transport T_k by

$$C_{ij} - D_{ij} \equiv T_{ij} = \tfrac{2}{3}\delta_{ij} T_k \tag{1}$$

Mellor and Yamada[4] also obtained Eq. (1) following a different analytical path. Rodi[1] proposed that

$$T_{ij} = \frac{\overline{u_i u_j}}{k} T_k \tag{2}$$

Both forms contract properly and display the correct symmetry in i and j. However, while Eq. (1) gives no transport of the off-diagonal elements (i.e., the shear stresses), Eq. (2) suggests that shear stress transport proceeds just as readily as for the normal stress components. The latter postulate seems more reasonable and has been used by Rodi,[1] Gibson and Launder,[5] and others in the computation of turbulent free shear flows. In these studies the Reynolds stress transport equation is approximated as

$$T_{ij} = P_{ij} - \tfrac{2}{3}\delta_{ij}\epsilon - c_1 \epsilon\left(\frac{\overline{u_i u_j}}{k} - \tfrac{2}{3}\delta_{ij}\right) - c_2\left(P_{ij} - \tfrac{1}{3}\delta_{ij} P_{kk}\right) \tag{3}$$

where ϵ is the dissipation rate of turbulence energy and P_{ij} the generation rate of $\overline{u_i u_j}$ by mean shear. A discussion of the physical basis of Eq. (3) is provided in Launder et al.[6]

Elimination of T_{ij} with Eq. (1) or (2) [noting that T_k equals $(\tfrac{1}{2}P_{kk} - \epsilon)$] allows Eq. (3) to be rearranged to an algebraic equation for $\overline{u_i u_j}$. For example, with Eq. (2) we obtain

$$\frac{\overline{u_i u_j} - \tfrac{2}{3}\delta_{ij} k}{k} = \frac{(1 - c_2)}{(c_1 - 1 + \lambda)}\left(P_{ij} - \frac{\delta}{3} ijP_{kk}\right)\Big/\epsilon \tag{4}$$

where λ denotes $P_{kk}/2\epsilon$ the local ratio of turbulent kinetic energy production to dissipation. For a two-dimensional thin shear flow (with x the flow direction and y the direction of principal gradient), Eq. (4) may be solved to give the following explicit formula for the shear stress

$$-\overline{uv} = \underbrace{\tfrac{2}{3}(1 - c_2)\frac{(c_1 - 1 + c_2\lambda)}{(c_1 - 1 + \lambda)^2}\frac{k^2}{\epsilon}}_{c_\mu(\lambda)}\frac{\partial U}{\partial y} \tag{5}$$

a form first given in Ref. 1.

The quantity c_μ in Eq. (5) increases as λ tends toward zero. Figure 1 shows the variation reported in Gibson and Launder[5] where the constant coefficients c_1 and c_2 were given the values 2.2 and 0.55, respectively. The upward trend is, however, less rapid than that exhibited by the experimental data of far wakes and the wakes of self-propelled bodies collected by Rodi[1] whose recommended mean line also appears in Fig. 1. A further unsatisfactory feature of Eq. (5) is that on the axis or plane of symmetry of a jet, λ falls to zero, yet here experiments suggest that a value of c_μ much closer to the local equilibrium ($\lambda = 1$) value is called for.

In considering the cause of these disparities, note first that the off-diagonal Reynolds stresses are predominantly associated with the largest eddies present in the flow and, thus, the turbulent kinetic energy is associated with rather

Table 1 Values of c_μ in round wake and plane jet

Flow	Experiment Min	Experiment Axis	Eq. (8) Min	Eq. (8) Axis	Eq. (5) Min	Eq. (5) Axis
Round wake	0.45	0.56	0.45	0.53	0.2	0.25
Plane jet	0.08	0.12	0.10	0.19	0.10	0.25

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